HARDER-NARASIMHAN FILTRATIONS AND K-GROUPS OF AN ELLIPTIC CURVE

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Abstract. Let X be an elliptic curve over an algebraically closed field. We prove that some exact sub-categories of the category of all vector bundles over X, defined using Harder-Narasimhan filtrations, have the same K-groups as the whole category.

1. Introduction

Throughout this paper, k denotes an algebraically closed field. Let X be a smooth projective curve over k and let E be a vector bundle over X. We define the slope of E as the quotient of its degree by its rank, i.e. $\mu(E) = deg(E)/rank(E)$. A vector bundle E is called semi-stable (resp. stable) if for any non-zero proper sub-bundle E', we have $\mu(E') \leq \mu(E)$ (resp. $\mu(E') < \mu(E)$). The importance of the notion of semi-stability consists in the constructions of moduli spaces of vector bundles, see for example [8][12][13][6][4]. For each vector bundle E, there exists a unique filtration, say *Harder-Narasimhan filtration* ([5, Proposition 1.3.9]),

$$0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_{s-1} \subsetneq E_s = E$$

such that the quotients $F_i = E_i/E_{i-1}$ are semi-stable for all $1 \leq i \leq s$ and

$$\mu(F_1) > \mu(F_2) > \cdots > \mu(F_s).$$

We note $\mu_{\max}(E) = \mu(F_1)$ and $\mu_{\min}(E) = \mu(F_s)$.

Let $\mathcal{P}(X)$ be the exact category of all vector bundles over X. Let $I \subset \mathbb{R}$ be a connected interval (possibly of length zero). Following T. Bridgeland([2, Section 3]), denote by $\mathcal{P}(I)$ the full sub-category of $\mathcal{P}(X)$ consisting of all vector bundles E such that $\mu_{\max}(E), \mu_{\min}(E) \in I$. It is an interesting fact that the category $\mathcal{P}(I)$ is also exact with the exact category structure induced from that of $\mathcal{P}(X)$ (see Lemma 2.1 below). We can therefore consider K-groups of $\mathcal{P}(I)$, as defined by D.Quillen for an exact category using his famous Q-construction ([11]). In this paper, we are interested in the relations between K-groups of $\mathcal{P}(I)$ and K-groups of $\mathcal{P}(X)$, i.e. those of X in case that X is an elliptic curve. More precisely, we prove the following theorem.

Theorem 1.1. *Let X be an elliptic curve over k an algebraically closed field and let I be a connected interval of strictly positive length. Then the inclusion functor $\mathcal{P}(I) \hookrightarrow \mathcal{P}(X)$ induces isomorphisms of K-groups $K_i(\mathcal{P}(I)) \xrightarrow{\cong} K_i(X)$ for all $i \geq 0$.*

Vector bundles over an elliptic curve were classified by M.Atiyah in [1]. His classification is essential to the proof of the preceding theorem. Roughly speaking, the idea is to construct, for an enough general vector bundle, a resolution of length one in $\mathcal{P}(I)$ and then the resolution theorem([11, Theorem 3.3]) applies.

The following question is natural.

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Question 1.2. *Does the statement in the preceding theorem hold if we replace X by any smooth projective curve of genus ≥ 2?*

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2. Proof of the main theorem

Firstly we prove the following fact mentioned in the introduction.

Lemma 2.1. *The category $\mathcal{P}(I)$ is an exact category whose exact sequences are given by short exact sequences in $\mathcal{P}(X)$ with their terms in $\mathcal{P}(I)$.*

Proof. One needs to show that $\mathcal{P}(I)$ is closed under extensions. Take a short exact sequence

$$0 \to E' \to E \to E'' \to 0$$

with $E', E'' \in \mathcal{P}(I)$. Let

$$0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_{s-1} \subsetneq E_s = E$$

be the Harder-Narasimhan filtration of E. We then have the exact sequence

$$0 \to E' \cap E_{s-1} \to E' \to F \to 0$$

with F a sub-bundle of $F_{s-1} = E/E_{s-1}$. We obtain that

$$\mu_{\min}(E) = \mu(F_{s-1}) \geq \mu(F) \geq \mu_{\min}(E').$$

We also have the exact sequence

$$0 \to E' \cap E_1 \to E_1 \to G \to 0$$

with G a sub-bundle of E''. We get that $\mu(E' \cap E_1) \leq \mu_{\max}(E')$ and $\mu(G) \leq \mu(E'') \leq \mu_{\max}(E'')$ and as $\mu_{\max}(E) = \mu(E_1)$ is the barycenter of $\mu(E' \cap E_1)$ and $\mu(G)$ with positive coefficients, $\mu_{max}(E) \leq \mu_{max}(E'), \mu_{max}(E'')$. This prove that $E \in \mathcal{P}(I)$.

□

Next we recall some known facts about vector bundles over an elliptic curve X.

Lemma 2.2. [6, Chapter 8, Section 8.7, Exercise 2.2] *Each vector bundle over X is a direct sum of indecomposable bundles. In particular, every indecomposable vector bundle is semi-stable.*

Theorem 2.3. *Let E and F be two semi-stable vector bundles over an elliptic curve. Then $E \otimes F$ is still semi-stable.*

In fact, in case of characteristic zero, the tensor product of two semi-stable vector bundles is semi-stable over a smooth projective curve of arbitrary genus. This was first proved by M.S.Narasimhan and C.S.Seshadri using analytic method ([9]) and then by Y. Miyaoka using algebraic method ([7, Corollary 3.7]). The case of positive characteristic uses the notion of strong semi-stability. A vector bundle is called strongly semi-stable if all its Frobenius pullbacks are semi-stable. T. Oda proved in [10, Theorem 2.16] (see also [14, Corollary 3^p]) that a semi-stable vector bundle over an elliptic curve is strongly semi-stable. Then the preceding theorem follows form the facts ([7, Section 5]) that the tensor product of two strongly semi-stable vector bundles is still strongly semi-stable and that strong semi-stability implies semi-stability.

Let $\mathcal{E}(r,d)$ with $r \geq 1$ and $d \in \mathbb{Z}$ be the set of isomorphism classes of indecomposable vector bundles of rank r and of degree d. When r and d are coprime, M. Atiyah introduced a distinguished vector bundle $E_{r,d} \in \mathcal{E}(r,d)$ (Atiyah noted it by $E_A(r,d)$) with the property $E_{r,d}^* \cong E_{r,-d}$ ([1, Corollary of Theorem 7]).

Let us construct the resolutions of length one for an enough general vector bundle. The starting point is the following lemma.

Lemma 2.4. *Let $E \in \mathcal{E}(r,d)$ with $r \geq 1$ and $d > 0$. Then there exists a vector bundle $E' \in \mathcal{E}(r+d,d)$, unique up to isomorphisms, given by the extension*

$$0 \to H^0(E) \otimes \mathcal{O}_X \to E' \to E \to 0$$

Moreover, $H^0(E) \cong H^0(E')$ and the map $H^0(E') \otimes \mathcal{O}_X \cong H^0(E) \otimes \mathcal{O}_X \to E'$ is the evaluation map.

Proof. The existence of E' follows from [1, Lemma 16] and other statements are easy consequences of [1, Lemma 15]. □

Proposition 2.5. *Let $E \in \mathcal{E}(r,d)$ with $r \geq 1$ and $d > 0$ and let $\epsilon > 0$. There exists a short exact sequence*

$$0 \to E_1 \to E_0 \to E \to 0$$

where E_1 is semi-stable of zero slope and where E_0 is semi-stable of slope $\mu(E_0) \in (0, \epsilon)$.

Proof. The preceding lemma gives an exact sequence

$$0 \to H^0(F_1) \otimes \mathcal{O}_X \xrightarrow{ev} F_1 \xrightarrow{f_1} E \to 0$$

with $F_1 \in \mathcal{E}(r+d,r)$ and where ev is the evaluation map. We again apply Lemma 2.4 to F_1 and we obtain

$$0 \to H^0(F_2) \otimes \mathcal{O}_X \xrightarrow{ev} F_2 \xrightarrow{f_2} F_1 \to 0$$

with $F_2 \in \mathcal{E}(r+2d,d)$. These two exact sequences yield

$$0 \to \mathrm{Ker}(f_1 \circ f_2) \to F_2 \xrightarrow{f_1 \circ f_2} E \to 0$$

and

$$0 \to H^0(F_2) \otimes \mathcal{O}_X \to \mathrm{Ker}(f_1 \circ f_2) \to H^0(F_1) \otimes \mathcal{O}_X \to 0 \quad (*)$$

Lemma 2.1 implies that $\mathrm{Ker}(f_1 \circ f_2)$ is semi-stable of zero slope.

If we iterate this process for n times with n enough great such that $d/(r+nd) < \epsilon$, we get

$$0 \to \mathrm{Ker}(f_1 \circ \cdots \circ f_n) \to F_n \xrightarrow{f_1 \circ \cdots \circ f_n} E \to 0 \quad (**).$$

As above, it is easy to show that $\mathrm{Ker}(f_n \circ \cdots \circ f_0)$ is semi-stable of zero slope and that (**) is the desired resolution.

□

Now we give the proof of the main theorem.

Proof. (of Theorem 1.1)

We can suppose that $I = (a,b)$ with $-\infty < a < b < +\infty$. For any real number λ, we note $I + \lambda = (a+\lambda, b+\lambda)$. Set $J = (a, +\infty)$.

Step I: *We show that the inclusion functor $\mathcal{P}(I) \hookrightarrow \mathcal{P}(J)$ induces isomorphisms of K-groups.* Take two integers $r \geq 1$ and d such that $-\frac{d}{r} = -\mu \in I$, $(r,d) = 1$ and $(r,p) = 1$ if $\mathrm{char} k = p > 0$. By Theorem 2.3, the tensor product by $E_{r,d}$ is an exact functor from $\mathcal{P}(I)$ to $\mathcal{P}(I+\mu)$. Note that $0 \in I + \mu$. Let $E \in \mathcal{P}(J)$. Then $E \otimes E_{r,d} \in \mathcal{P}((a+\mu, +\infty))$. Suppose that $E \otimes E_{r,d} = \oplus F_i$ with all F_i indecomposable. If $F_i \in \mathcal{P}(I+\mu)$, then we take the resolution

$$0 \to 0 \to F_i \xrightarrow{Id} F_i \to 0$$

and if $F_i \notin \mathcal{P}(I+\mu)$, we take the resolution given by Proposition 2.5 with $\epsilon = b+\mu$. The sum of these resolutions of all F_i is a resolution of E of the form

$$0 \to E_1 \to E_0 \to E \otimes E_{r,d} \to 0$$

where E_1 is semi-stable of zero slope and where E_0 is in $\mathcal{P}(I+\mu)$. Now the tensor product of the resolution above by $E_{r,-d}$ gives

$$0 \to E_1 \otimes E_{r,-d} \to E_0 \otimes E_{r,-d} \xrightarrow{f} E \otimes E_{r,d} \otimes E_{r,-d} \to 0$$

By [10, Corollary 2.7], $E_{r,d} \otimes E_{r,-d} \cong \mathcal{E}nd(E_{r,d}) = \mathcal{O}_X \oplus G$. We write g the projection from $E \otimes E_{r,d} \otimes E_{r,-d}$ to $E \otimes G$. We have an exact sequence

$$0 \to E_1 \otimes E_{r,-d} \to \mathrm{Ker}(g \circ f) \to E \to 0$$

Obviously $E_1 \otimes E_{r,-d}$ is semi-stable of slope $-\mu$. The inequality $\mu_{max}(\mathrm{Ker}(g \circ f)) \leq \mu_{max}(E_0 \otimes E_{r,-d})$ together with Lemma 2.1 implies that $\mathrm{Ker}(g \circ f) \in \mathcal{P}(I)$. The resolution theorem applies and we obtain that the inclusion functor $\mathcal{P}(I) \hookrightarrow \mathcal{P}(J)$ induces isomorphisms of K-groups.

Step II: *We show that the inclusion functor $\mathcal{P}(J) \hookrightarrow \mathcal{P}(X)$ induces isomorphisms of K-groups.* By a theorem of Serre ([3, Chapter 2, Theorem 5.17]), for each $E \in \mathcal{P}(X)$, we have an exact sequence

$$0 \to E \to \mathcal{O}_X(n)^m \to F \to 0$$

with $n, m >> 0$ and then $\mathcal{O}_X(n)^m, F \in \mathcal{P}(J)$. Let us consider the functor $\mathcal{P}(J)^{op} \hookrightarrow \mathcal{P}(X)^{op}$ where *op* means the opposite category. Notice that $Q\mathcal{C}^{op} \cong Q\mathcal{C}$([11, Page 94]) where Q is the Q-construction and then $K_i(\mathcal{C}^{op}) \cong K_i(\mathcal{C})$ for all $i \geq 0$, we can deduce from the resolution theorem that the inclusion functor $\mathcal{P}(J) \hookrightarrow \mathcal{P}(X)$ induces isomorphisms of K-groups.

This finishes the proof.

□

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